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The Lebesgue Constants for Cardinal Spline Interpolation*

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If $(y_\nu) \in l_\infty$, let $\mathscr{L}_n y$ be the unique bounded cardinal spline of degree $n-1$ interpolating to y at the integers, i.e.,

$$\mathscr{L}_n y(\nu) = y_\nu, \; \nu = 0, \pm 1, \pm 2.$$

The norm of this operator: $\| \mathscr{L}_n \| = \sup \| \mathscr{L}_n y \| / \| y \|$ is called a Lebesgue constant. A formula for $\| \mathscr{L}_n \|$ is obtained, and with it we will show that

$$\| \mathscr{L}_n \| = \frac{2}{\pi} \log n + \frac{2}{\pi}\left(2 \log \frac{4}{\pi} + \gamma\right) + o(1) \quad \text{as} \quad n \to \infty$$

where γ is the Euler–Mascheroni constant.

1. INTRODUCTION

If n is a natural number, let us define the space $\mathscr{S}_n = \{S(x)\}$ of bounded cardinal splines of degree $n-1$ to consist of those functions satisfying the following conditions:

(i) $S \in C^{n-2}(-\infty, \infty)$,

(ii) $\| S \| = \sup_{-\infty < x < \infty} | S(x) | < \infty$,

(iii) $S(x)$ reduces to a polynomial of degree at most $n-1$ on each of the intervals $[\nu + n/2, \nu + n/2 + 1]$, $\nu = 0, \pm 1, \pm 2, \ldots$, i.e., $S(x)$ has knots at the integers or half integers if n is, respectively, even or odd.

If $y = (y_\nu)_{\nu=-\infty}^{\infty} \in l_\infty$, the space of (real or complex) doubly infinite bounded sequences, then there is a unique element $\mathscr{L}_n y \in \mathscr{S}_n$ interpolating the given data at the integers, i.e.,

$$\mathscr{L}_n y(\nu) = y_\nu, \qquad \nu = 0, \pm 1, \pm 2, \ldots .$$

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The operator $\mathscr{L}_n: l_\infty \to \mathscr{S}_n$ is called the cardinal spline interpolation operator of order n, and its norm

$$\|\mathscr{L}_n\| = \sup_{\|y\|_\infty = 1} \|\mathscr{L}_n y\|,$$

is referred to as the nth Lebesgue constant for cardinal spline interpolation. These numbers have been investigated previously for low values of n (see [2, 6–8]). The purpose of this paper is to examine the asymptotic behavior of the Lebesgue constants as the degree becomes large. More specifically, the following result is obtained.

THEOREM 1. *Let γ be the Euler–Mascheroni constant. Then*

$$\lim_{n\to\infty} \left\{\|\mathscr{L}_n\| - \frac{2}{\pi}\log n\right\} = \frac{2}{\pi}\left(2\log\frac{4}{\pi} + \gamma\right) = 0.675.... \tag{1.1}$$

It seems of interest to compare this with results obtained for polynomial interpolation operators. Let

$$\Delta_n : -1 \leqslant x_n^n < x_{n-1}^n < \cdots < x_1^n \leqslant 1, \qquad n = 1, 2,..., \tag{1.2}$$

be a given infinite triangular array, and for each $f \in C[-1, 1]$, let $\mathscr{P}_{\Delta_n} f$ denote the unique polynomial of degree less than n satisfying

$$\mathscr{P}_{\Delta_n} f(x_\nu^n) = f(x_\nu^n), \qquad \nu = 1,..., n.$$

Erdös [1] has shown that there exists a constant c independent of the array (1.2) such that

$$\|\mathscr{P}_{\Delta_n}\| \geqslant (2/\pi)\log n - c. \tag{1.3}$$

On the other hand, if

$$\bar{\Delta}_n : x_\nu^n = \cos\frac{(2\nu - 1)\pi}{2n}, \qquad \nu = 1, 2,..., n,$$

are the zeros of the nth Chebyshev polynomial, Rivlin [3] has shown that

$$\lim_{n\to\infty} \left\{\|\mathscr{P}_{\bar{\Delta}_n}\| - \frac{2}{\pi}\log n\right\} = \frac{2}{\pi}\left(\log\frac{8}{\pi} + \gamma\right) = 0.9625.... \tag{1.4}$$

Hence we make the surprising observation that "near-best" polynomial interpolation Lebesgue constants display nearly the same asymptotic behavior as cardinal spline interpolation Lebesgue constants.

2. A Formula for $\|\mathscr{L}_n\|$.

We now discuss certain functions and concepts that will play a major role in our discussion. Define

$$\psi_n(t) = \left(\frac{2 \sin t/2}{t}\right)^n \tag{2.1}$$

and

$$\varphi_n(t) = \sum_{j=-\infty}^{\infty} \psi_n(t + 2\pi j). \tag{2.2}$$

Then the Fourier transform of $\psi_n(t)/\varphi_n(t)$ is

$$L_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi_n(t)}{\varphi_n(t)} e^{itx}\, dt, \tag{2.3}$$

which is characterized by the properties (see [4])

(i) $L_n \in \mathscr{S}_n$,

(ii) $L_n(\nu) = \begin{cases} 1, & \nu = 0, \\ 0, & \nu = \pm 1, \pm 2, \ldots, \end{cases}$

(iii) $|L_n(x)| \to 0$ exponentially as $|x| \to \infty$,

i.e., $L_n(x)$ is the "fundamental" cardinal spline of degree $n - 1$. Thus

$$(\mathscr{L}_n y)(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_n(x - \nu) \qquad -\infty < x < \infty. \tag{2.4}$$

Since

$$|\mathscr{L}_n y(x)| \leqslant \|y\|_\infty \sum_{\nu=-\infty}^{\infty} |L_n(x - \nu)|$$

and $\sum_{\nu=-\infty}^{\infty} |L_n(x - \nu)|$ has period 1, it is clear that

$$\|\mathscr{L}_n\| \leqslant \max_{0 \leqslant x \leqslant 1} \sum_{\nu=-\infty}^{\infty} |L_n(x - \nu)|\,.$$

On the other hand, if we assume $n \geqslant 3$ and consider the sequence

$$\tilde{y}_\nu = \operatorname{sgn} L_n(x - \nu) = \begin{cases} (-1)^{\nu+1}, & \nu = 1, 2, \ldots, \\ (-1)^\nu, & \nu = 0, -1, -2, \ldots, \end{cases} \qquad 0 < x < 1. \tag{2.5}$$

then

$$\|\mathscr{L}_n \tilde{y}\| = \sup_{-\infty < x < \infty} \sum_{\nu=-\infty}^{\infty} \tilde{y}_\nu L_n(x - \nu) = \max_{0 \leqslant x \leqslant 1} \sum_{\nu=-\infty}^{\infty} |L_n(x - \nu)| \tag{2.6}$$

and the maximum occurs at $x = \frac{1}{2}$ (see [2]). Thus

$$\| \mathscr{L}_n \| = \sum_{\nu=-\infty}^{\infty} \tilde{y}_\nu L_n(\tfrac{1}{2} - \nu). \tag{2.7}$$

The following theorem gives a more useful formula for $\| \mathscr{L}_n \|$.

THEOREM 2. *Define the function,*

$$\gamma_n(t) = \sum_{j=-\infty}^{\infty} (-1)^j \psi_n(t + 2\pi j). \tag{2.8}$$

Then if $n \geqslant 3$, *we have*

$$\| \mathscr{L}_n \| = \frac{1}{\pi} \int_0^\pi \frac{\gamma_n(t)}{\varphi_n(t)} \sec \frac{t}{2}\, dt\,. \tag{2.9}$$

Proof. By (2.3),

$$\sum_{\nu=-N+1}^{N} \tilde{y}_\nu L_n \left(\frac{1}{2} - \nu\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{\psi_n(t)}{\varphi_n(t)} e^{it/2} \sum_{\nu=-N+1}^{N} \tilde{y}_\nu e^{-it\nu} \right\} dt. \tag{2.10}$$

But

$$\sum_{\nu=-N+1}^{N} \tilde{y}_\nu e^{-it\nu} = -\sum_{\nu=1}^{N} (-e^{-it})^\nu + \sum_{\nu=0}^{N-1} (-e^{it})^\nu$$

are just geometric series. An easy calculation yields

$$\sum_{\nu=-N+1}^{N} \tilde{y}_\nu e^{-it\nu} = \frac{2}{1 + e^{it}} (1 + (-1)^{N+1} \cos Nt). \tag{2.11}$$

Plugging into (2.10) we obtain

$$\begin{aligned} \sum_{\nu=-N+1}^{N} \tilde{y}_\nu L_n \left(\frac{1}{2} - \nu\right) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\psi_n(t)}{\varphi_n(t)} \sec \frac{t}{2} (1 + (-1)^{N+1} \cos Nt)\, dt \\ &= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \int_{2\pi j - \pi}^{2\pi j + \pi} \frac{\psi_n(t)}{\varphi_n(t)} \sec \frac{t}{2} (1 + (-1)^{N+1} \cos Nt)\, dt \\ &= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{(-1)^j \psi_n(t + 2\pi j)}{\varphi_n(t)} \sec \frac{t}{2} (1 + (-1)^{N+1} \cos Nt)\, dt, \end{aligned} \tag{2.12}$$

by the periodicity of $\varphi_n(t)$ and the fact that $\sec((t + 2\pi j)/2) = (-1)^j \sec t/2$. But since $\gamma_n(t) = \sum_{j=-\infty}^{\infty} (-1)^j \psi_n(t + 2\pi j)$ converges uniformly and the resulting integrand is an even function, it follows that

$$\sum_{\nu=-N+1}^{N} \tilde{y}_\nu L_n\left(\frac{1}{2} - \nu\right)$$
$$= \frac{1}{\pi} \int_0^\pi \frac{\gamma_n(t)}{\varphi_n(t)} \sec \frac{t}{2}\, dt + \frac{(-1)^{N+1}}{2\pi} \int_{-\pi}^{\pi} \frac{\gamma_n(t)}{\varphi_n(t)} \sec \frac{t}{2} \cos Nt\, dt. \qquad (2.13)$$

It will be shown in the next section that $[\gamma_n(t)/\varphi_n(t)] \sec t/2$ is continuous on $[-\pi, \pi]$. Hence letting $N \to \infty$ in (2.13), the second term on the right will $\to 0$ by the Riemann–Lebesgue lemma, and the term on the left will $\to \| \mathscr{L}_n \|$ by (2.7). This proves (2.9).

3. The Functions $\varphi_n(t)$ and $\gamma_n(t)$

In order to use formula (2.9) to prove Theorem 1, it will be necessary to simplify the integrand. Thus we must examine in more detail the functions $\varphi_n(t)$ and $\gamma_n(t)$. Consider the functions

$$\rho_n(t) = \left(2 \sin \frac{t}{2}\right)^n \sum_{j=-\infty}^{\infty} \frac{1}{(t + 2\pi j)^n} \qquad (3.1)$$

and

$$\sigma_n(t) = \left(2 \sin \frac{t}{2}\right)^n \sum_{j=-\infty}^{\infty} \frac{(-1)^j}{(t + 2\pi j)^n}. \qquad (3.2)$$

Then

$$\varphi_n(t) = \left(2 \sin \frac{t}{2}\right)^n \sum_{j=-\infty}^{\infty} \frac{(-1)^{jn}}{(t + 2\pi j)^n} = \begin{cases} \rho_n(t) & n \text{ even}, \\ \sigma_n(t) & n \text{ odd}, \end{cases} \qquad (3.3)$$

and

$$\gamma_n(t) = \left(2 \sin \frac{t}{2}\right)^n \sum_{j=-\infty}^{\infty} \frac{(-1)^{j(n+1)}}{(t + 2\pi j)^n} = \begin{cases} \sigma_n(t) & n \text{ even}, \\ \rho_n(t) & n \text{ odd}. \end{cases} \qquad (3.4)$$

Schoenberg [4, 5] has shown that $\rho_n(t)$ and $\sigma_n(t)$ are trigonometric polynomials in the variable $t/2$ and that $\varphi_n(t) > 0$, $-\infty < t < \infty$. From (3.4) it follows that $\gamma_n(t) > 0$, $-\pi < t < \pi$, and $\gamma_n(\pi) = \gamma_n(-\pi) = 0$. Thus $[\gamma_n(t)/\varphi_n(t)] \sec t/2$ is indeed a continuous, nonnegative function on $[-\pi, \pi]$, as claimed in §2.

The following lemma states that it is "all right" to replace the functions $\gamma_n(t)$ and $\varphi_n(t)$ in (2.9) by their "dominant" terms.

LEMMA 1. *If $n \geqslant 3$ and $0 < t < \pi$, then*

$$\left| \frac{\varphi_n(t)}{\psi_n(t) + \psi_n(t - 2\pi)} - 1 \right| < 3^{-n+1} \tag{3.5}$$

and

$$\left| \frac{\gamma_n(t)}{\psi_n(t) - \psi_n(t - 2\pi)} - 1 \right| < 2^{-n}. \tag{3.6}$$

Proof. Consider the functions

$$\begin{aligned} \tilde{\varphi}_n(u) &= \sum_{j=-\infty}^{\infty} \frac{(-1)^{jn}}{(u+j)^n} \\ &= \frac{1}{u^n} + \frac{1}{(1-u)^n} + \sum_{j=1}^{\infty} (-1)^{jn} \left[\frac{1}{(j+u)^n} + \frac{1}{(j+1-u)^n} \right] \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \tilde{\gamma}_n(u) &= \sum_{j=-\infty}^{\infty} \frac{(-1)^{j(n+1)}}{(u+j)^n} \\ &= \frac{1}{u^n} - \frac{1}{(1-u)^n} + \sum_{j=1}^{\infty} (-1)^{j(n+1)} \left[\frac{1}{(j+u)^n} - \frac{1}{(j+1-u)^n} \right] \end{aligned} \tag{3.8}$$

By (3.3) and (3.4), it is seen that

$$\sup_{0<t<\pi} \left| \frac{\varphi_n(t)}{\psi_n(t) + \psi_n(t - 2\pi)} - 1 \right| = \sup_{0<u<\frac{1}{2}} \left| \frac{\tilde{\varphi}_n(u)}{1/u^n + 1/(1-u)^n} - 1 \right|$$

and

$$\sup_{0<t<\pi} \left| \frac{\gamma_n(t)}{\psi_n(t) - \psi_n(t - 2\pi)} - 1 \right| = \sup_{0<u<\frac{1}{2}} \left| \frac{\tilde{\gamma}_n(u)}{1/u^n - 1/(1-u)^n} - 1 \right|.$$

The fact that $0 < u < \frac{1}{2}$ will be used repeatedly. We compute

$$\begin{aligned} \left| \frac{\tilde{\varphi}_n(u)}{1/u^n + (1/(1-u)^n)} - 1 \right| &\leqslant \frac{\sum_{j=1}^{\infty} [1/(j+u)^n + 1/(j+1-u)^n]}{1/u^n + 1/(1-u)^n} \\ &\leqslant \frac{\sum_{j=1}^{\infty} (u/(j+u))^n + (u/(j+1-u))^n}{1 + (u/(1-u))^n} \\ &\leqslant 2 \sum_{j=1}^{\infty} (2j+1)^{-n} \leqslant 3^{-n+1} \quad \text{for} \quad n \geqslant 3. \end{aligned}$$

Thus (3.5) is proved. From (3.8) we have

$$\left| \frac{\tilde{\gamma}_n(u)}{1/u^n - 1/(1-u)^n} - 1 \right| \leqslant \sum_{j=1}^{\infty} \alpha_j(u, n),$$

where

$$0 \leqslant \alpha_j(u, n) = \frac{1/(j+u)^n - 1/(j+1-u)^n}{1/u^n - 1/(1-u)^n}.$$

It follows easily that

$$\alpha_j(u, n) \leqslant \frac{u^n[(j+1-u)^n - (j+u)^n]}{(j+u)^n (j+1-u)^n (1 - u^2/(1-u)^2)},$$

and since

$$\frac{(j+1-u)^n - (j+u)^n}{1-2u} = \sum_{i=0}^{n-1} (j+1-u)^{n-1-i} (j+u)^i \leqslant n(j+1-u)^{n-1},$$

we get

$$\alpha_j(u, n) \leqslant \frac{u^n(1-u)^2\, n}{(j+u)^n (j+1-u)} \leqslant n \left(\frac{u}{j+u}\right)^n \frac{1}{(j+1-u)} \leqslant \frac{2n}{(2j+1)^{n+1}}.$$

Therefore,

$$\sum_{j=1}^{\infty} \alpha_j(u, n) \leqslant 2n \cdot 3^{-n-1} + 3^{-n} < 2^{-n}, \qquad \text{for} \quad n \geqslant 3.$$

This establishes (3.6).

Let

$$R_n(t) = \frac{\gamma_n(t)[\psi_n(t) + \psi_n(t-2\pi)]}{\varphi_n(t)[\psi_n(t) - \psi_n(t-2\pi)]}, \qquad 0 < t < \pi. \tag{3.9}$$

An immediate consequence of Lemma 1 and (2.9) is

LEMMA 2. *If $n \geqslant 3$, there exists a point $\xi_n \in [0, \pi]$ such that*

$$\| \mathscr{L}_n \| = \frac{R_n(\xi_n)}{\pi} \int_0^{\pi} \frac{(2\pi - t)^n - t^n}{(2\pi - t)^n + t^n} \sec \frac{t}{2}\, dt \tag{3.10}$$

and

$$| R_n(\xi_n) - 1 | < 2^{-n+2}. \tag{3.11}$$

Thus if the sequence

$$\| \mathscr{L}_n \|^* = \frac{1}{\pi} \int_0^{\pi} \frac{(2\pi - t)^n - t^n}{(2\pi - t)^n + t^n} \sec \frac{t}{2}\, dt, \qquad n = 3, 4, \ldots, \tag{3.12}$$

is $o(2^n)$, then

$$\| \mathscr{L}_n \|^* - \| \mathscr{L}_n \| \to 0, \qquad \text{as} \qquad n \to \infty.$$

4. Proof of Theorem 1

Formula (3.12) will now be used to prove Theorem 1. Since

$$\sec \frac{t}{2} = \frac{2}{\pi - t} + h(t), \qquad 0 < t < \pi,$$

where $h(t) \in C[0, \pi]$, (3.12) gives us

$$\begin{aligned} \| \mathscr{L}_n \|^* &= \frac{2}{\pi} \int_0^{\pi} \frac{(2\pi - t)^n - t^n}{(2\pi - t)^n + t^n} \frac{dt}{\pi - t} + \frac{1}{\pi} \int_0^{\pi} \frac{(2\pi - t)^n - t^n}{(2\pi - t)^n + t^n} h(t)\, dt \\ &= A_n + B_n . \end{aligned} \tag{4.1}$$

The second integrand remains bounded and converges a.e. to $h(t)$ as $n \to \infty$. Thus by bounded convergence,

$$\lim_{n \to \infty} B_n = \frac{1}{\pi} \int_0^{\pi} h(t)\, dt = \frac{2}{\pi} \log \frac{4}{\pi} . \tag{4.2}$$

Dividing the numerator and denominator of the first integrand in (4.1) by t^n and making the change of variables $t \to x = 2\pi/t - 1$ we obtain

$$A_n = \frac{4}{\pi} \int_1^{\infty} \frac{(x^n - 1)}{(x^n + 1)} \frac{dx}{(x^2 - 1)} . \tag{4.3}$$

Assume for the time being that n is even, i.e., $n = 2m$. Then

$$A_{2m} = \frac{4}{\pi} \int_1^{\infty} \frac{x^{2m-2} + x^{2m-4} + \cdots + 1}{x^{2m} + 1}\, dx.$$

Since $\int_0^1 = \int_1^{\infty}$ and the integrand is even, one gets

$$A_{2m} = \sum_{\nu=1}^{m} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x^{2m-2\nu}}{x^{2m} + 1}\, dx. \tag{4.4}$$

Each of these integrals may be evaluated by complex integration as follows. Let $R > 1$ and consider the contour proceeding from $z = -R$ to $z = R$ along the real axis, and then back to $z = -R$ along the upper half of a circle of radius R centered at the origin. The integrands

$$z^{2m-2\nu}/(z^{2m}+1), \qquad \nu = 1, 2,..., m$$

have only simple poles inside the contour at the points

$$z_r = e^{\pi i(2r-1)/2m}, \qquad r = 1, 2,..., m$$

and corresponding residues

$$\frac{z_r^{1-2\nu}}{2m} = \frac{1}{2m} e^{-\pi i(2r-1)(2\nu-1)/2m}.$$

Summing and letting $R \to \infty$ we see that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^{2m-2\nu}}{x^{2m}+1}\,dx = \frac{1}{m\sin((\pi/2m)(2\nu-1))}. \tag{4.5}$$

Hence A_{2m} is a Riemann sum for the divergent integral $(1/\pi)\int_0^\pi \csc x\,dx$.

Now $\csc x - 1/x - 1/(\pi - x)$ is a continuous and hence Riemann integrable function on $[0, \pi]$ and

$$\int_0^\pi \left(\csc x - \frac{1}{x} - \frac{1}{\pi - x}\right) dx = 2\log\frac{2}{\pi}. \tag{4.6}$$

Using (4.4) and (4.5) and replacing the resulting Riemann sum by means of (4.6) we have

$$\begin{aligned} A_{2m} &= \frac{2}{m}\sum_{\nu=1}^{m} \frac{2m}{\pi(2\nu-1)} + \frac{2}{\pi}\log\frac{2}{\pi} + o(1) \\ &= \frac{4}{\pi}\left\{\sum_{\nu=1}^{2m}\frac{1}{\nu} - \frac{1}{2}\sum_{\nu=1}^{m}\frac{1}{\nu}\right\} + \frac{2}{\pi}\log\frac{2}{\pi} + o(1) \\ &= \frac{2}{\pi}\log 2m + \frac{2}{\pi}\left(\log\frac{4}{\pi} + 2\gamma_{2m} - \gamma_m\right) + o(1). \end{aligned} \tag{4.7}$$

Here

$$\gamma_n = \sum_{\nu=1}^{n}\frac{1}{\nu} - \log n, \qquad n = 1, 2,...,$$

and $\gamma_n \to \gamma$, the Euler–Mascheroni constant. Now observe that the integrand of (4.3) is an *increasing* function of n, and hence $A_{n+1} > A_n$. Thus the estimate (4.7) is valid for *all* $n \geqslant 3$. Using (4.1) we obtain

$$\| \mathscr{L}_n \|^* = \frac{2}{\pi} \log n + \frac{2}{\pi} \left(2 \log \frac{4}{\pi} + \gamma\right) + o(1). \tag{4.8}$$

After considering the remark at the end of the last section, we see that we may replace $\| \mathscr{L}_n \|^*$ by $\| \mathscr{L}_n \|$ in (4.8). This concludes the proof of Theorem 1.

In concluding the paper, we state without proof a result concerning the monotonicity of the sequence $\| \mathscr{L}_n \|$, $n = 1, 2, \ldots$.

THEOREM 3.

$$1 = \| \mathscr{L}_1 \| = \| \mathscr{L}_2 \| < \| \mathscr{L}_3 \| < \| \mathscr{L}_4 \| < \cdots.$$

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